9.5 - Rotation of Conics/General Form of Conics

Classifying a conic if if the conic is not rotated

The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is one of the following:

1. Circle A = C, A≠0
2. Parabola AC = 0, A=0 or C=0 but not both
3. Ellipse AC > 0, A and C have like signs
4. Hyperbola AC < 0, A and C have unlike signs

Ex. 1 - Classify the following conic sections by examining their general equations:

a. $4x^2 - 9x + y - 5 = 0$ ____

b. $4x^2 - y^2 + 8x - 6y + 4 = 0$ ____________________________________

c. $2x^2 + 4y^2 - 4x + 12y = 0$ _____________________________________

d. $2x^2 + 2y^2 - 8x + 12y + 2 = 0$ __________________________________

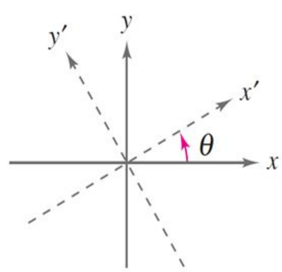
The General form of an equation of a conic with an axes parallel to either the x or y axis:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

The General form of an equation of a conic whose <u>axes are rotated</u> so that they are not parallel to either the *x*-axis or the *y*-axis. The general equation for such conics <u>contains an *xy*-term.</u>

$$Ax^2 + \underline{Bxy} + Cy^2 + Dx + Ey + F = 0.$$

To eliminate this *xy*-term, you can use a procedure called **rotation of axes.** The objective is to rotate the *x*- and *y*-axes until they are parallel to the axes of the conic. The rotated axes are denoted as the <u>*x* ' axis</u> and the <u>*y* ' axis</u>



After the rotation, the equation of the conic in the new *x ' y '* - plane will have the form:

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0.$$

Because this equation has no *xy*- term, *you can* obtain a standard form by completing the square.

Rotation of Axes to Eliminate an *xy*-Term

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

The coefficients of the new equation are obtained by making the substitutions

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta.$$

- When trying to find θ (the angle that the x and y axes are rotated through), there are 2 possibilities to consider:

1. If cot (2θ) = 0, then 2θ = π/2, which means θ = π/4
2. If cot (2θ) > 0, then 0° < 2θ < 90°, so 0° < θ < 45°
3. If cot (2θ) < 0, then 90° < 2θ < 180°, so 45° < θ < 90°

**arccot 2θ has a similar range as arccos 2θ , which is 0° < θ < 180° (Quad 1 If cot (2θ) > 0, Quad 2 If cot (2θ) < 0)

** If cot (2θ) ≠ 0 , first find cos (2θ). Then use the inverse cosine function key to obtain the value of 2θ, where 0° < 2θ < 180°. Finally, divide by 2 to obtain the correct angle θ.

Ex.2 - Solve for θ, when 0° ≤ θ < 360°

1. cot (2θ) = 1 ____

2. cot (2θ) = $\sqrt{3}$ ____

3. $\cot(2\theta) = -\frac{3}{4}$ ____

4. $\cot(2\theta) = -7/24$ ____

Ex.3 - Determine the appropriate rotation formulas to use so that the new equation contains no xy-term.

$x^2 - 4xy + y^2 - 3 = 0$ ____

1. 1cot (2θ) = $\frac{A-C}{B} = \frac{1-1}{-4} = 0$, if cot (2θ) = 0 , then 2θ = π/2, thus <u>θ = π/4</u>

2. $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$
   $x = x'\cos(\pi/4) - y'\sin(\pi/4) = x'\frac{\sqrt{2}}{2} - y'\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(x' - y')$
   $y = x'\sin(\pi/4) + y'\cos(\pi/4) = x'\frac{\sqrt{2}}{2} + y'\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(x' + y')$

A. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$ ____

B. $x^2 + 4xy + 4y^2 + 5\sqrt{5}y + 5 = 0$ ____

C. $34x^2 - 24xy + 41y^2 - 25 = 0$ ____

Ex.4 - Rotate the axes so that the new equation contains no xy-term. Discuss and graph the new equation by hand.

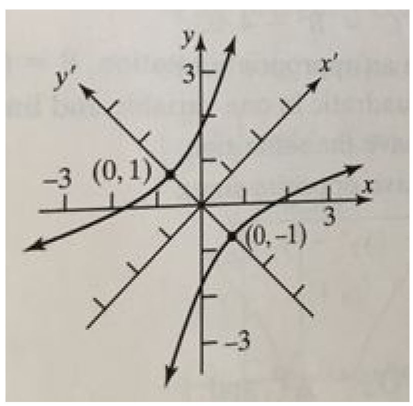
x^2 - 4xy + y^2 - 3 = 0 ____

1. cot (2θ) = $\frac{A-C}{B}$ = $\frac{1-1}{-4}$ = 0 , if cot (2θ) = 0 , then 2θ = π/2, thus θ = π/4

2. x = x' cosθ - y'sinθ and y = x' sinθ + y'cosθ
x = x' cos(π/4) - y'sin(π/4) = x'$\frac{\sqrt{2}}{2}$ - y'$\frac{\sqrt{2}}{2}$ = $\frac{\sqrt{2}}{2}$(x' - y')
y= x' sin(π/4) + y'cos(π/4) = x'$\frac{\sqrt{2}}{2}$ + y'$\frac{\sqrt{2}}{2}$ = $\frac{\sqrt{2}}{2}$(x' + y')

3. Substitute these expressions in for x and y of the original equation
$[\frac{\sqrt{2}}{2}(x' - y')]^2$ - 4 $[\frac{\sqrt{2}}{2}(x' - y')][\frac{\sqrt{2}}{2}(x' + y')]$ + $[\frac{\sqrt{2}}{2}(x' + y')]^2$ = 3
$\frac{(y')^2}{1} - \frac{(x')^2}{3} = 1$

4. This equation yields a hyperbola with a center at (0,0), a TVA axis parallel to the Y'-axis, vertices at (0, 1) and (0,-1) on the y' axis. (because a = 1).
The graph on the x'y' - axis is then rotated π/4 units



A. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$ ____

B. $x^2 + 4xy + 4y^2 + 5\sqrt{5}y + 5 = 0$ ____

C. $34x^2 - 24xy + 41y^2 - 25 = 0$ ____

- **Identifying conics without a rotation of axes**

  The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ defines:

  a. A parabola if $B^2 - 4AC = 0$

  b. An ellipse if $B^2 - 4AC < 0$

  c. A hyperbola if $B^2 - 4AC > 0$

Ex.5 - Identify each equation without applying a rotation of axes.

A. $2x^2 - 3xy + 4y^2 + 2x + 3y - 5 = 0$

Since $B^2 - 4AC = (-3)^2 - 4(2)(4) = \underline{-23 < 0}$, thus this conic is an <u>ellipse</u> since A ≠ C

B. $10x^2 + 12xy + 4y^2 - x - y + 10 = 0$

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C. $4x^2 + 12xy + 9y^2 - x - y = 0$

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D. $3x^2 + 2xy + y^2 + 4x - 2y + 10 = 0$ ____